Vector Analyses of Propagation Constants in Dielectric Optical Waveguides with Perturbed Refractive-Index Profile

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Abstract—**A new method is developed to evaluate the propagation constants of guided or leaky modes in cylindrical dielectric optical waveguides with arbitrary cross sections. It is assumed that the refractive-index profiles of the fiber deviate from the step-index ones. Formal expressions of the propagation constants are given for waveguides with anisotropically perturbed refractive-index profiles. The method is applied to a circular step-index fiber with an azimuthally perturbed refractive-index profile, and it is shown that scalar analyses cannot properly predict the propagation constants of those modes whose angular order satisfies certain conditions with respect to the Fourier spectrum of the perturbation of the refractive-index profile.**

I. Introduction

As FABRICATION techniques and applications are maturing, optical fibers are being made with various cross sections and refractive-index profiles. In order to compute the propagation properties of these fibers, to design fibers with prescribed properties, or to investigate the effects of waveguide imperfections, several methods of analysis are generally available, based on scalar or vector theories as summarized in [1]. Analytical methods, if they can be successfully applied, have the advantage of yielding explicit expressions for the propagation constants of modes and, therefore, of their group delays.

When the refractive-index difference between the core region and the cladding is small, the difference of the propagation constants predicted by the scalar and vector analyses seems to be small [2]–[5]. Naturally, polarization properties of the modes have been discussed vectorially [4], [6]–[7]. There is no exact analytical treatment for fibers that have azimuthally perturbed refractive-index profiles [8], [9], although some vector analyses [5], [7] are available for fibers with only radially perturbed ones.

On the other hand, a coupled-mode theory [10], [11] is often used to analyze anisotropically perturbed optical fibers [12]. By noticing that the coupled-mode theory itself is based on the completeness of the modes, one cannot generally ensure the results in fibers which support a few modes when the radiation modes are not taken into account.

In this paper, we present a new analytical method to evaluate the propagation constants of the guided modes in circular step-index fibers with anisotropically, azimuthally, radially perturbed refractive-index profiles. The new method requires only lower order field distributions to evaluate propagation constants when they are expanded in a series of a perturbation term. The method is extended to waveguides with arbitrary cross sections and also to waveguides which support leaky modes. An essential difference between results predicted by the present method and by the scalar analysis is shown to exist for circular step-index fibers with an azimuthally perturbed refractive-index profile.

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II. General Expression of Propagation Constants in Cylindrical Waveguides with Perturbed Refractive-Index Profile

A. Circular Waveguides with Anisotropical Perturbation

We first consider a circular waveguide whose refractive index $n_0(r)$ has a staircase distribution as shown in Fig. 1. When the optical fiber suffers from perturbations such as an elliptical core, elastic deformations, or a bent axis, the refractive index $n(r,\theta)$ is expressed by

$$n^2(r,\theta) = n_0^2(r) + 2\epsilon\Delta(r,\theta) \tag{1}$$

where ϵ is a small perturbation parameter and $\Delta(r,\theta)$ (or simply written Δ) is a tensor expressed as

$$\Delta = \begin{pmatrix} \Delta_{rr} & \Delta_{r\theta} & \Delta_{rz} \\ \Delta_{\theta r} & \Delta_{\theta\theta} & \Delta_{\theta z} \\ \Delta_{zr} & \Delta_{z\theta} & \Delta_{zz} \end{pmatrix}. \tag{2}$$

Δ_{ij} $(i,j = r,\theta,z)$ are functions of the transverse coordinates r and θ.

By expanding the electric and magnetic fields E and H and the propagation constant β of the guided modes as [13]

$$E = E^{(0)} + \epsilon E^{(1)} + \epsilon^2 E^{(2)} + \cdots \tag{3}$$

$$H = H^{(0)} + \epsilon H^{(1)} + \epsilon^2 H^{(2)} + \cdots \tag{4}$$

$$\beta = \beta^{(0)} + \epsilon\beta^{(1)} + \epsilon^2\beta^{(2)} + \cdots \tag{5}$$

and substituting (3)–(5) into Maxwell's equations

$$\frac{1}{r}\frac{\partial H_z}{\partial\theta} + j\beta H_\theta = j\omega\epsilon_0\left[n_0^2(r)E_r + 2\epsilon\Delta E\cdot\hat{r}\right] \tag{6}$$

$$-j\beta H_r - \frac{\partial H_z}{\partial r} = j\omega\epsilon_0\left[n_0^2(r)E_\theta + 2\epsilon\Delta E\cdot\hat{\theta}\right] \tag{7}$$

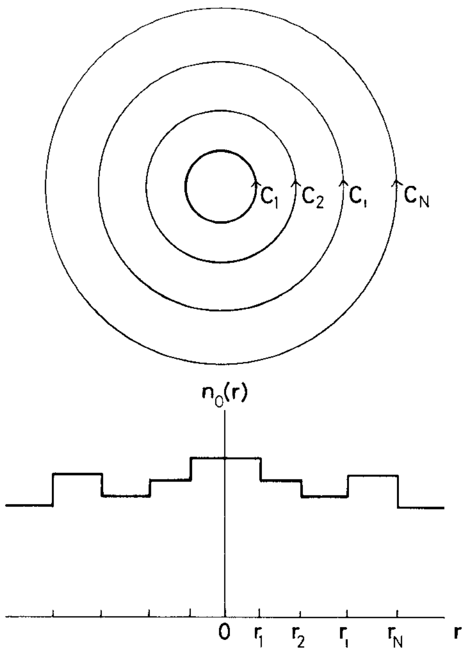

Fig. 1. Refractive-index profile in a circular step-index fiber with the refractive-index discontinuity at $r = r_1, r_2, \cdots, r_N$. Path integrals appearing in the text should be done counterclockwise as shown in the figure.

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = j\omega\epsilon_0\left[n_0^2(r)E_z + 2\epsilon\Delta E\cdot\hat{z}\right] \quad (8)$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} + j\beta E_\theta = -j\omega\mu_0 H_r \quad (9)$$

$$-j\beta E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu_0 H_\theta \quad (10)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -j\omega\mu_0 H_z \quad (11)$$

we can obtain transverse electric and magnetic fields $E_r^{(i)}$, $E_\theta^{(i)}$, $H_r^{(i)}$, and $H_\theta^{(i)}$ $(i = 0,1,2)$ from (6), (7), (9), and (10) as follows:

$$\begin{aligned}\kappa_0^2(r)E_r^{(i)} = {} & \beta^{(2)}\left[\beta^{(0)}E_r^{(i-2)} + \omega\mu_0 H_\theta^{(i-2)}\right] \\ & + \beta^{(1)}\left[\beta^{(0)}E_r^{(i-1)} + \omega\mu_0 H_\theta^{(i-1)}\right] \\ & - 2k_0^2\Delta E^{(i-1)}\cdot\hat{r} \\ & - j\left[\beta^{(0)}\frac{\partial E_z^{(i)}}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial H_z^{(i)}}{\partial \theta}\right]\end{aligned} \quad (12)$$

$$\begin{aligned}\kappa_0^2(r)E_\theta^{(i)} = {} & \beta^{(2)}\left[\beta^{(0)}E_\theta^{(i-2)} - \omega\mu_0 H_r^{(i-2)}\right] \\ & + \beta^{(1)}\left[\beta^{(0)}E_\theta^{(i-1)} - \omega\mu_0 H_r^{(i-1)}\right] \\ & - 2k_0^2\Delta E^{(i-1)}\cdot\hat{\theta} \\ & - j\left[\frac{\beta^{(0)}}{r}\frac{\partial E_z^{(i)}}{\partial \theta} - \omega\mu_0\frac{\partial H_z^{(i)}}{\partial r}\right]\end{aligned} \quad (13)$$

$$\begin{aligned}\kappa_0^2(r)H_r^{(i)} = {} & \beta^{(2)}\left[\beta^{(0)}H_r^{(i-2)} - \omega\epsilon_0 n_0^2(r)E_\theta^{(i-2)}\right] \\ & + \beta^{(1)}\left[\beta^{(0)}H_r^{(i-1)} - \omega\epsilon_0 n_0^2(r)E_\theta^{(i-1)}\right] \\ & + 2\omega\epsilon_0\beta^{(0)}\Delta E^{(i-1)}\cdot\hat{\theta} \\ & + j\left[\frac{\omega\epsilon_0 n_0^2(r)}{r}\frac{\partial E_z^{(i)}}{\partial \theta} - \beta^{(0)}\frac{\partial H_z^{(i)}}{\partial r}\right]\end{aligned} \quad (14)$$

$$\begin{aligned}\kappa_0^2(r)H_\theta^{(i)} = {} & \beta^{(2)}\left[\beta^{(0)}H_\theta^{(i-2)} + \omega\epsilon_0 n_0^2(r)E_r^{(i-2)}\right] \\ & + \beta^{(1)}\left[\beta^{(0)}H_\theta^{(i-1)} + \omega\epsilon_0 n_0^2(r)E_r^{(i-1)}\right] \\ & - 2\omega\epsilon_0\beta^{(0)}\Delta E^{(i-1)}\cdot\hat{r} \\ & - j\left[\omega\epsilon_0 n_0^2(r)\frac{\partial E_z^{(i)}}{\partial r} + \frac{\beta^{(0)}}{r}\frac{\partial H_z^{(i)}}{\partial \theta}\right]\end{aligned} \quad (15)$$

where $\hat{r}$ and $\hat{\theta}$ are unit vectors along the r and θ directions, respectively, $\kappa_0^2(r)$ is defined by

$$\kappa_0^2(r) = n_0^2(r)k_0^2 - \beta^{(0)^2} \quad (16)$$

and quantities with negative superscripts are understood to be zero.

By substituting (14) and (15) or (12) and (13) into (8) or (11), one obtains the differential equations to determine $E_z^{(i)}$ or $H_z^{(i)}$ $(i = 0,1,2)$ as follows:

$$\begin{aligned}\nabla^2 E_z^{(i)} + \kappa_0^2(r)E_z^{(i)} & \\ = {} & \left[2\beta^{(0)}\beta^{(2)} + \beta^{(1)^2}\right]E_z^{(i-2)} + 2\beta^{(0)}\beta^{(1)}E_z^{(i-1)} \\ & + j\frac{2}{n_0^2(r)}\nabla\cdot\left[\beta^{(0)}\Delta E^{(i-1)} + \beta^{(1)}\Delta E^{(i-2)}\right] \\ & - 2\frac{\kappa_0^2(r)}{n_0^2(r)}\Delta E^{(i-1)}\cdot\hat{z} + 4\frac{\beta^{(0)}\beta^{(1)}}{n_0^2(r)}\Delta E^{(i-2)}\cdot\hat{z}\end{aligned} \quad (17)$$

$$\begin{aligned}\nabla^2 H_z^{(i)} + \kappa_0^2(r)H_z^{(i)} & \\ = {} & \left[2\beta^{(0)}\beta^{(2)} + \beta^{(1)^2}\right]H_z^{(i-2)} + 2\beta^{(0)}\beta^{(1)}H_z^{(i-1)} \\ & - j2\omega\epsilon_0\nabla\times\left[\Delta E^{(i-1)}\right]\cdot\hat{z}\end{aligned} \quad (18)$$

where ∇^2, $\nabla\cdot$, and $\nabla\times$ are two-dimensional operators. No restricted assumptions have yet been made to derive (12)–(18) from Maxwell's equations.

We now explain how to express the first-order correction term $\beta^{(1)}$ to $\beta^{(0)}$ in terms of the known zeroth-order electric and magnetic fields $E^{(0)}$ and $H^{(0)}$.

Forming

$$\omega\epsilon_0 n_0^2(r)\left[E_z^{(0)}\nabla^2 E_z^{(1)} - E_z^{(1)}\nabla^2 E_z^{(0)}\right] \quad (19)$$

with the help of (17), and integrating over the innermost region $r < r_1$ (see Fig. 1), one obtains

$$\begin{aligned}\beta^{(1)}\Big\{ & 2\beta^{(0)}\int_{S_1}\omega\epsilon_0 n_0^2(r)E_z^{(0)^2}\,dS \\ & + j\oint_{C_1}\left[\beta^{(0)}H_\theta^{(0)} + \omega\epsilon_0 n_0^2(r)E_r^{(0)}\right]E_z^{(0)}\,dC\Big\} \\ = {} & 2\int_{S_1}\omega\epsilon_0 n_0^2(r)G_E^{(0)}E_z^{(0)}\,dS + j2\omega\epsilon_0\beta^{(0)}\oint_{C_1}\Delta E^{(0)}\cdot\hat{r}E_z^{(0)}\,dC \\ & + j\oint_{C_1}\kappa_0^2(r)\left[E_z^{(0)}H_\theta^{(1)} - H_\theta^{(0)}E_z^{(1)}\right]dC \\ & + \beta^{(0)}\oint_{C_1}\frac{1}{r}\left[E_z^{(1)}\frac{\partial H_z^{(0)}}{\partial \theta} - E_z^{(0)}\frac{\partial H_z^{(1)}}{\partial \theta}\right]dC\end{aligned} \quad (20)$$

where C_1 denotes the periphery $r = r_1$, and $G_E^{(0)}$ is defined by

$$G_E^{(0)} = \frac{\kappa_0^2(r)}{n_0^2(r)} \Delta E^{(0)} \cdot \hat{z} - j \frac{\beta^{(0)}}{n_0^2(r)} \nabla \cdot [\Delta E^{(0)}]. \quad (21)$$

To derive (20), Green's theorem was used to transform the surface integral into the line integral around the periphery $r = r_1$.

Similarly, integrating $\omega\mu_0[H_z^{(0)} \nabla^2 H_z^{(1)} - H_z^{(1)} \nabla^2 H_z^{(0)}]$ in the same region, one obtains

$$\beta^{(1)} \left\{ 2\beta^{(0)} \int_{S_1} \omega\mu_0 H_z^{(0)^2} \, dS - j \oint_{C_1} [\beta^{(0)} E_\theta^{(0)} - \omega\mu_0 H_r^{(0)}] H_z^{(0)} \, dC \right\}$$
$$= 2 \int_{S_1} \omega\mu_0 G_H^{(0)} H_z^{(0)} \, dS - j2k_0^2 \oint_{C_1} \Delta E^{(0)} \cdot \hat{\theta} H_z^{(0)} \, dC$$
$$+ j \oint_{C_1} \kappa_0^2(r) [E_\theta^{(0)} H_z^{(1)} - E_\theta^{(1)} H_z^{(0)}] \, dC$$
$$+ \beta^{(0)} \oint_{C_1} \frac{1}{r} \left[H_z^{(0)} \frac{\partial E_z^{(1)}}{\partial \theta} - H_z^{(1)} \frac{\partial E_z^{(0)}}{\partial \theta} \right] dC \quad (22)$$

where $G_H^{(0)}$ is defined by

$$G_H^{(0)} = j\omega\epsilon_0 \nabla \times [\Delta E^{(0)}] \cdot \hat{z}. \quad (23)$$

By adding (20) and (22) and noticing that

$$\beta^{(0)} \oint_{C_1} \frac{1}{r} \left[E_z^{(1)} \frac{\partial H_z^{(0)}}{\partial \theta} - E_z^{(0)} \frac{\partial H_z^{(1)}}{\partial \theta} \right] dC + \beta^{(0)} \oint_{C_1} \frac{1}{r} \left[H_z^{(0)} \frac{\partial E_z^{(1)}}{\partial \theta} - H_z^{(1)} \frac{\partial E_z^{(0)}}{\partial \theta} \right] dC$$
$$= \beta^{(0)} \oint_{C_1} \frac{1}{r} \frac{\partial}{\partial \theta} [E_z^{(1)} H_z^{(0)} - E_z^{(0)} H_z^{(1)}] \, dC$$
$$= 0 \quad (24)$$

one obtains

$$\beta^{(1)} \left\{ 2\beta^{(0)} \int_{S_1} [\omega\epsilon_0 n_0^2(r) E_z^{(0)^2} + \omega\mu_0 H_z^{(0)^2}] \, dS + j \oint_{C_1} [\beta^{(0)} H_\theta^{(0)} + \omega\epsilon_0 n_0^2(r) E_r^{(0)}] E_z^{(0)} \, dC - j \oint_{C_1} [\beta^{(0)} E_\theta^{(0)} - \omega\mu_0 H_r^{(0)}] H_z^{(0)} \, dC \right\}$$
$$= 2 \int_{S_1} [\omega\epsilon_0 n_0^2(r) G_E^{(0)} E_z^{(0)} + \omega\mu_0 G_H^{(0)} H_z^{(0)}] \, dS$$
$$+ j2 \oint_{C_1} [\omega\epsilon_0 \beta^{(0)} \Delta E^{(0)} \cdot \hat{r} E_z^{(0)} - k_0^2 \Delta E^{(0)} \cdot \hat{\theta} H_z^{(0)}] \, dC$$
$$+ j \oint_{C_1} \kappa_0^2(r) [E_z^{(0)} H_\theta^{(1)} - H_\theta^{(0)} E_z^{(1)} + E_\theta^{(0)} H_z^{(1)} - E_\theta^{(1)} H_z^{(0)}] \, dC. \quad (25)$$

One should note that the first-order fields appear in the third term on the right-hand side.

In order to eliminate this term, we first use similar integrals over annular regions defined by radii $r = r_1$ and r_2, r_2 and $r_3, \cdots$, and also over the infinite region from $r = r_N$ to ∞. By dividing each equation thus obtained by $\kappa_0^2(r)$, which is piecewise constant, and summing up all equations, we eliminate the integrals where the first-order fields appear. The line integrals appear twice in opposite directions at the discontinuity $r = r_i$ $(i = 1, 2, \cdots, N)$ and the line integral along $r = \infty$ becomes zero because all fields decay exponentially. Therefore, the final expression for $\beta^{(1)}$ becomes

$$\beta^{(1)} \left\{ 2\beta^{(0)} \int \frac{1}{\kappa_0^2(r)} [\omega\epsilon_0 n_0^2(r) E_z^{(0)^2} + \omega\mu_0 H_z^{(0)^2}] \, dS + j \sum_{i=1}^{N} \oint_{C_i} \delta\left[\frac{1}{\kappa_0^2(r_i)}\right] [\beta^{(0)} H_\theta^{(0)} + \omega\epsilon_0 n_0^2(r) E_r^{(0)}] E_z^{(0)} \, dC - j \sum_{i=1}^{N} \oint_{C_i} \delta\left[\frac{1}{\kappa_0^2(r_i)}\right] [\beta^{(0)} E_\theta^{(0)} - \omega\mu_0 H_r^{(0)}] H_z^{(0)} \, dC \right\}$$
$$= 2 \int \frac{1}{\kappa_0^2(r)} [\omega\epsilon_0 n_0^2(r) G_E^{(0)} E_z^{(0)} + \omega\mu_0 G_H^{(0)} H_z^{(0)}] \, dS$$
$$+ j \frac{2k_0^2}{\omega\mu_0} \sum_{i=1}^{N} \oint_{C_i} \left\{ \beta^{(0)} E_z^{(0)} \delta\left[\frac{\Delta E^{(0)} \cdot \hat{r}}{\kappa_0^2(r_i)}\right] - \omega\mu_0 H_z^{(0)} \delta\left[\frac{\Delta E^{(0)} \cdot \hat{\theta}}{\kappa_0^2(r_i)}\right] \right\} dC \quad (26)$$

where $\delta[f(r_i, \theta)]$ stands for the discontinuity at $r = r_i$ defined by

$$\delta[f(r_i, \theta)] = f(r_i - 0, \theta) - f(r_i + 0, \theta). \quad (27)$$

When $\beta^{(1)} = 0$, we can proceed to obtain the correction term $\beta^{(2)}$ by a method similar to the one described above. In this case, $E_z^{(2)}$ and $H_z^{(2)}$ satisfy

$$\nabla^2 E_z^{(2)} + \kappa_0^2(r) E_z^{(2)} = 2\beta^{(0)}\beta^{(2)} E_z^{(0)} - 2G_E^{(1)} \quad (28)$$
$$\nabla^2 H_z^{(2)} + \kappa_0^2(r) H_z^{(2)} = 2\beta^{(0)}\beta^{(2)} H_z^{(0)} - 2G_H^{(1)} \quad (29)$$

where $G_E^{(1)}$ and $G_H^{(1)}$ are defined by

$$G_E^{(1)} = \frac{\kappa_0^2(r)}{n_0^2(r)} \Delta E^{(1)} \cdot \hat{z} - j \frac{\beta^{(0)}}{n_0^2(r)} \nabla \cdot [\Delta E^{(1)}] \quad (30)$$

$$G_H^{(1)} = j \frac{k_0^2}{\omega\mu_0} \nabla \times [\Delta E^{(1)}] \cdot \hat{z}. \quad (31)$$

Therefore, by integrating the following quantity over the whole transverse cross section

$$\frac{1}{\kappa_0^2(r)} \{ \omega\epsilon_0 n_0^2(r) [E_z^{(0)} \nabla^2 E_z^{(2)} - E_z^{(2)} \nabla^2 E_z^{(0)}] + \omega\mu_0 [H_z^{(0)} \nabla^2 H_z^{(2)} - H_z^{(2)} \nabla^2 H_z^{(0)}] \} \quad (32)$$

we finally express $\beta^{(2)}$ in terms of known quantities $\beta^{(0)}$,

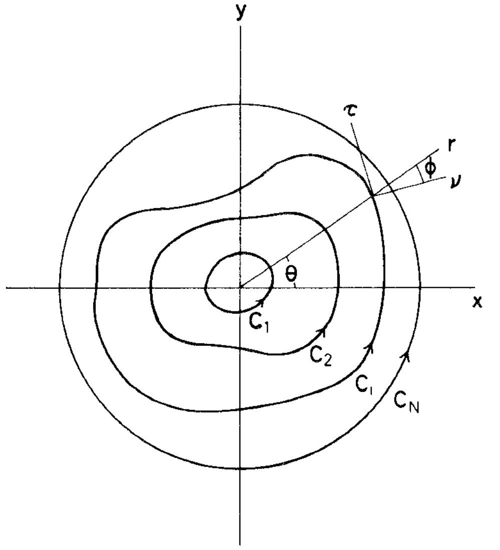

Fig. 2. The coordinate system (r, θ) and (ν, τ) in cylindrical waveguides with arbitrary cross section. ν and τ are perpendicular and parallel to each boundary C_i where the refractive index has a discontinuity. ϕ is the angle between r and ν.

$E^{(0)}$, $E^{(1)}$, $H^{(0)}$, and $H^{(1)}$ as follows:

$$\beta^{(2)}\left\{2\beta^{(0)}\int\frac{1}{\kappa_0^2(r)}\left[\omega\epsilon_0 n_0^2(r)E_z^{(0)^2}+\omega\mu_0 H_z^{(0)^2}\right]dS\right.$$
$$+j\sum_{i=1}^{N}\oint_{C_i}\delta\left[\frac{1}{\kappa_0^2(r_i)}\right]\left[\beta^{(0)}H_\theta^{(0)}+\omega\epsilon_0 n_0^2(r)E_r^{(0)}\right]E_z^{(0)}\,dC$$
$$\left.-j\sum_{i=1}^{N}\oint_{C_i}\delta\left[\frac{1}{\kappa_0^2(r_i)}\right]\left[\beta^{(0)}E_\theta^{(0)}-\omega\mu_0 H_r^{(0)}\right]H_z^{(0)}\,dC\right\}$$
$$=2\int\frac{1}{\kappa_0^2(r)}\left[\omega\epsilon_0 n_0^2(r)G_E^{(1)}E_z^{(0)}+\omega\mu_0 G_H^{(1)}H_z^{(0)}\right]dS$$
$$+j\frac{2k_0^2}{\omega\mu_0}\sum_{i=1}^{N}\oint_{C_i}\left\{\beta^{(0)}E_z^{(0)}\delta\left[\frac{\Delta E^{(1)}\cdot\hat{r}}{\kappa_0^2(r_i)}\right]\right.$$
$$\left.-\omega\mu_0 H_z^{(0)}\delta\left[\frac{\Delta E^{(1)}\cdot\hat{\theta}}{\kappa_0^2(r_i)}\right]\right\}dC. \tag{33}$$

One should note that determination of $\beta^{(1)}$ or $\beta^{(2)}$ does not require the evaluation of the perturbed fields to the same order, which simplifies the calculations considerably compared with an ordinary characteristic equation method.

B. *Cylindrical Waveguides with Arbitrary Cross Section*

Consider a waveguide with an arbitrary cross section as shown in Fig. 2, where the refractive indices of regions surrounded by closed curves $[C_i, C_{i+1}]$ $(i=1,2,\cdots,N-1)$ are constant. The circular coordinate system (r,θ) and the coordinate system (ν,τ) perpendicular and parallel to the boundaries are also used. Let the angle between ν and r be ϕ. One can express E_ν and E_τ at the boundaries in terms of E_r and E_θ of the circular coordinate system by

$$\begin{pmatrix}E_\nu\\E_\tau\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}E_r\\E_\theta\end{pmatrix}. \tag{34}$$

For an arbitrary scalar function F, we also obtain

$$\begin{pmatrix}\dfrac{\partial F}{\partial\nu}\\[2mm]\dfrac{\partial F}{\partial\tau}\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\dfrac{\partial F}{\partial r}\\[2mm]\dfrac{1}{r}\dfrac{\partial F}{\partial\theta}\end{pmatrix} \tag{35}$$

at the boundaries.

By using a procedure similar to that given in [14], one can extend the results (26) and (33) for circular waveguides to any cylindrical waveguides as follows:

$$\beta^{(1)}\left\{2\beta^{(0)}\int\frac{1}{\kappa_0^2}\left[\omega\epsilon_0 n_0^2 E_z^{(0)^2}+\omega\mu_0 H_z^{(0)^2}\right]dS\right.$$
$$+j\sum_{i=1}^{N}\oint_{C_i}\delta\left[\frac{1}{\kappa_0^2}\right]\left[\beta^{(0)}H_\tau^{(0)}+\omega\epsilon_0 n_0^2 E_\nu^{(0)}\right]E_z^{(0)}\,dC$$
$$\left.-j\sum_{i=1}^{N}\oint_{C_i}\delta\left[\frac{1}{\kappa_0^2}\right]\left[\beta^{(0)}E_\tau^{(0)}-\omega\mu_0 H_\nu^{(0)}\right]H_z^{(0)}\,dC\right\}$$
$$=2\int\frac{1}{\kappa_0^2}\left[\omega\epsilon_0 n_0^2 G_E^{(0)}E_z^{(0)}+\omega\mu_0 G_H^{(0)}H_z^{(0)}\right]dS$$
$$+j\frac{2k_0^2}{\omega\mu_0}\sum_{i=1}^{N}\oint_{C_i}\left\{\beta^{(0)}E_z^{(0)}\delta\left[\frac{\Delta E^{(0)}\cdot\hat{\nu}}{\kappa_0^2}\right]\right.$$
$$\left.-\omega\mu_0 H_z^{(0)}\delta\left[\frac{\Delta E^{(0)}\cdot\hat{\tau}}{\kappa_0^2}\right]\right\}dC \tag{36}$$

$$\beta^{(2)}\left\{2\beta^{(0)}\int\frac{1}{\kappa_0^2}\left[\omega\epsilon_0 n_0^2 E_z^{(0)^2}+\omega\mu_0 H_z^{(0)^2}\right]dS\right.$$
$$+j\sum_{i=1}^{N}\oint_{C_i}\delta\left[\frac{1}{\kappa_0^2}\right]\left[\beta^{(0)}H_\tau^{(0)}+\omega\epsilon_0 n_0^2 E_\nu^{(0)}\right]E_z^{(0)}\,dC$$
$$\left.-j\sum_{i=1}^{N}\oint_{C_i}\delta\left[\frac{1}{\kappa_0^2}\right]\left[\beta^{(0)}E_\tau^{(0)}-\omega\mu_0 H_\nu^{(0)}\right]H_z^{(0)}\,dC\right\}$$
$$=2\int\frac{1}{\kappa_0^2}\left[\omega\epsilon_0 n_0^2 G_E^{(1)}E_z^{(0)}+\omega\mu_0 G_H^{(1)}H_z^{(0)}\right]dS$$
$$+j\frac{2k_0^2}{\omega\mu_0}\sum_{i=1}^{N}\oint_{C_i}\left\{\beta^{(0)}E_z^{(0)}\delta\left[\frac{\Delta E^{(1)}\cdot\hat{\nu}}{\kappa_0^2}\right]\right.$$
$$\left.-\omega\mu_0 H_z^{(0)}\delta\left[\frac{\Delta E^{(1)}\cdot\hat{\tau}}{\kappa_0^2}\right]\right\}dC \tag{37}$$

where $\hat{\nu}$ and $\hat{\tau}$ are unit vectors perpendicular and parallel to each boundary, respectively.

In deriving (26) and (33) or (36) and (37), we have assumed that the fields decay exponentially at $r=\infty$ for guided modes and, therefore, we can delete the line integrals along $r=\infty$. For leaky modes, the situation is different from that of guided modes, i.e., the field intensities grow exponentially. However, by choosing a suitable path of integration at $r=\infty$ in the complex plane [15], [16], one can change the growing fields into decaying fields so that the line integral at $r=\infty$ also vanishes. Therefore,

(36) and (37) are shown to be applicable to leaky as well as guided modes.

III. Propagation Constant in Circular Dielectric Optical Waveguides with Azimuthally Perturbed Refractive-Index Profile

In this section, we apply the vector analysis given above to a simple case of a weakly guiding circular step-index fiber with an azimuthally perturbed refractive-index profile, and we compare the result with that predicted by a scalar analysis.

Let the core radius be T, and the refractive index of the core be n_0. We assume a perturbation of the form

$$\Delta(r,\theta)=p(r)\cos(q\theta) \tag{38}$$

which is an angular spectrum of the perturbed refractive-index profile. For simplicity, we consider a waveguide where the power of the guided mode is highly confined within the core. Therefore, we can put $E_r=E_\theta=H_r=H_\theta=0$ at $r=T$, and (26) is simplified as follows:

$$\beta^{(1)}\beta^{(0)}\int\left[\omega\epsilon_0 n_0^2 E_z^{(0)^2}+\omega\mu_0 H_z^{(0)^2}\right]dS$$
$$=k_0^2\int\Delta(r,\theta)\left[\omega\epsilon_0 n_0^2 E_z^{(0)^2}+\omega\mu_0 H_z^{(0)^2}\right]dS$$
$$+jk_0^2\int\left\{\frac{\partial\Delta}{\partial r}\left[E_\theta^{(0)}H_z^{(0)}-H_\theta^{(0)}E_z^{(0)}\right]\right.$$
$$\left.-\frac{1}{r}\frac{\partial\Delta}{\partial\theta}\left[E_r^{(0)}H_z^{(0)}-H_r^{(0)}E_z^{(0)}\right]\right\}dS \tag{39}$$

where the integral should be done in $r\leqslant T$, and we have used the relation $\beta^{(0)}\simeq n_0k_0$.

By substituting zeroth-order fields [13], one can easily deduce that the HE_{n+1m} and EH_{n-1m} modes are degenerate for $n\geqslant 2$, and the TE_{0m}, TM_{0m}, and HE_{2m} modes are degenerate. Therefore, by combining these circular modes properly, one can construct the LP_{nm} modes whose $\beta^{(1)}$ is evaluated as follows:

$$\beta^{(1)}=\frac{2k_0^2}{\beta_0T^2}\frac{\int_0^T J_n^2(ur/T)p(r)r\,dr}{J_{n-1}^2(u)}\delta_{q0} \tag{40}$$

where δ_{q0} is Kronecker's delta and u is the root of $J_n(u)=0$.

On the other hand, we can obtain $\beta^{(1)}$ for the scalar equation of

$$\nabla^2\psi+\left[k_0^2n^2(r,\theta)-\beta^2\right]\psi=0 \tag{41}$$

satisfying the boundary condition $\psi(T,\theta)=0$ as follows (see Appendix I):

$$\beta^{(1)}=\frac{2k_0^2}{\beta_0T^2}\frac{\int_0^T J_n^2(ur/T)p(r)r\,dr}{J_{n-1}^2(u)}\times\begin{cases}\cos 2\theta_0\delta_{2nq} & (q\neq 0)\\ 1 & (q=0)\end{cases} \tag{42}$$

where θ_0 represents the orientation of the mode, by which the zeroth-order solution $\psi^{(0)}$ is expressed as

$$\psi^{(0)}=J_n\left(u\frac{r}{T}\right)\cos(n\theta+\theta_0). \tag{43}$$

One should note that the scalar analysis cannot predict $\beta^{(1)}$ of the LP_{nm} mode whose angular order satisfies $2n=q$, i.e., the propagation constant predicted by the scalar analyses depends on the orientation.

By similar calculations for $\beta^{(2)}$ based on the scalar analysis (Appendix I), one can deduce that $\beta^{(2)}$ depends on the orientation when $n=q$ [17]. However, this fact does not agree with the result given in [13] even if the power of the mode is fully confined in the core. In fact, when $\Delta(r,\theta)$ takes the form

$$\Delta(r,\theta)=r\cos(\theta) \tag{44}$$

in a weakly guiding fiber whose power is fully confined in the core, $\beta^{(1)}=0$ and the present vector analysis gives $\beta^{(2)}$ of the LP_{nm} mode as follows [13]:

$$\beta^{(2)}=\frac{1}{12}\left(\frac{k_0T}{n_0u}\right)^2T^2\left[1-\frac{4(n^2-1)}{u^2}\right]. \tag{45}$$

On the other hand, the scalar analysis leads to (Appendix II)

$$\beta^{(2)}=\frac{1}{12}\left(\frac{k_0T}{n_0u}\right)^2T^2\left[1-\frac{4(n^2-1)}{u^2}-\frac{3}{2}\cos 2\theta_0\delta_{n1}\right] \tag{46}$$

which shows that it does not give the correct result for the LP_{1m} mode. Notice that if we put

$$\epsilon=\frac{n_0^2}{R} \tag{47}$$

$\epsilon^2\beta^{(2)}$ corresponds to the phase constant change of the mode due to a uniform waveguide bend of the bending radius R.

IV. Conclusion

A new analytical method is presented to calculate the propagation constants of the guided as well as leaky modes in cylindrical step-index fibers with anisotropically, azimuthally, radially perturbed refractive-index profiles. The method is applied to a simple case, and it is shown that scalar analyses cannot properly predict the propagation constants of particular modes.

Further application of the method will be given by emphasizing the relation with the coupled-mode analysis.

Appendix I

Consider the following scalar wave equation:

$$\nabla^2\psi+\left[k_0^2n^2(r,\theta)-\beta^2\right]\psi=0 \tag{A1}$$

where $n^2(r,\theta)$ is represented by

$$n^2(r,\theta)=n_0^2(r,\theta)+2\epsilon\Delta(r,\theta) \tag{A2}$$

and $n_0^2(r,\theta)$ is a piecewise step function in a transverse cross section.

By expanding ψ as

$$\psi=\psi^{(0)}+\epsilon\psi^{(1)}+\epsilon^2\psi^{(2)}+\cdots \tag{A3}$$

we can obtain differential equations for $\psi^{(i)}$ $(i=0,1,2)$ as follows:

$$\nabla^2\psi^{(0)}+\left[k_0^2n_0^2(r,\theta)-\beta^{(0)^2}\right]\psi^{(0)}=0 \tag{A4}$$

$$\nabla^2\psi^{(1)}+\left[k_0^2n_0^2(r,\theta)-\beta^{(0)^2}\right]\psi^{(1)} = 2\left[\beta^{(0)}\beta^{(1)}-k_0^2\Delta(r,\theta)\right]\psi^{(0)} \tag{A5}$$

$$\nabla^2\psi^{(2)}+\left[k_0^2n_0^2(r,\theta)-\beta^{(0)^2}\right]\psi^{(2)}=\left[2\beta^{(0)}\beta^{(2)}+\beta^{(1)^2}\right]\psi^{(0)} +2\left[\beta^{(0)}\beta^{(1)}-k_0^2\Delta(r,\theta)\right]\psi^{(1)}. \tag{A6}$$

By following similar procedures in Section II-A and noticing that ψ and $\partial\psi/\partial\nu$ are continuous at each discontinuity of the refractive index, we can express $\beta^{(1)}$ and $\beta^{(2)}$ as follows:

$$\beta^{(1)}=\frac{k_0^2\int\Delta(r,\theta)\psi^{(0)^2}\,dS}{\beta^{(0)}\int\psi^{(0)^2}\,dS} \tag{A7}$$

$$\beta^{(2)}=\frac{\int\left[k_0^2\Delta(r,\theta)-\beta^{(0)}\beta^{(1)}\right]\psi^{(0)}\psi^{(1)}\,dS}{\beta^{(0)}\int\psi^{(0)^2}\,dS}-\frac{\beta^{(1)^2}}{2\beta^{(0)}} \tag{A8}$$

where the integral should be done over the whole cross-sectional area. One should note that (A7) is exactly the same as that given in [18].

Appendix II

We assume a weakly guiding circular fiber with the refractive index of the core n_0 and with core radius T. By setting

$$\Delta(r,\theta)=r\cos(\theta) \tag{A9}$$

and solving (A4) and (A5) under the condition that $\psi^{(0)}(r,\theta)=\psi^{(1)}(r,\theta)=0$ at $r=T$, one obtains

$$\psi^{(0)}=J_n\left(u\frac{r}{T}\right)\cos(n\theta+\theta_0) \tag{A10}$$

$$\psi^{(1)}=\frac{(k_0T)^2T}{4u}\left\{g_n\left(\frac{r}{T}\right)\cos\left[(n+1)\theta+\theta_0\right] - h_n\left(\frac{r}{T}\right)\cos\left[(n-1)\theta+\theta_0\right]\right\} \tag{A11}$$

where

$$g_n(x)=J_{n+1}(ux)+x^2J_{n-1}(ux) \tag{A12}$$

$$h_n(x)=J_{n-1}(ux)+x^2J_{n+1}(ux) \tag{A13}$$

and u is the root of $J_n(u)=0$.

By substituting (A10) and (A11) into (A8) with $\beta^{(1)}=0$, one finally obtains (46).

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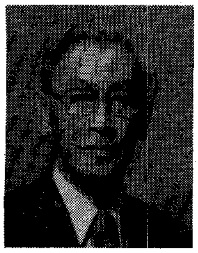

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